King Fahd University of Petroleum and Minerals
College of Computer Sciences and Engineering
Information and Computer Science Department

ICS 253: Discrete Structures I
Summer semester 2017-2018
Major Exam #2, Thursday August 2, 2018
Time: **100** Minutes

Name: ____

ID#: ________________________________

Instructions:

1. This exam consists of **8** pages, including this page and an additional separate helping sheet, containing **six** questions.

2. Answer all **six** questions. *Show all the steps*.

3. Make sure your answers are **clear** and **readable**.

4. The exam is closed book and closed notes. No calculators or any helping aides are allowed. Make sure you turn off your mobile phone and keep it in your pocket.

5. If there is no space on the front of the page, use the back of the page.

Question	Maximum Points	Earned Points
1	25	
2	20	
3	10	
4	10	
5	15	
6	20	
Total	**100**	

Q1: [25 points] Answer the following questions.

a) (10 points) Give an example of two uncountable sets A and B such that $A \cap B$ is

i. Empty.

$A = (0,1)$, $B = (2,3)$

ii. Finite but not empty.

$A = (0,1]$, $B = [1,2)$

iii. Countably infinite.

$A = (0,1) \cup \mathbb{Z}$
$B = (2,3) \cup \mathbb{Z}$

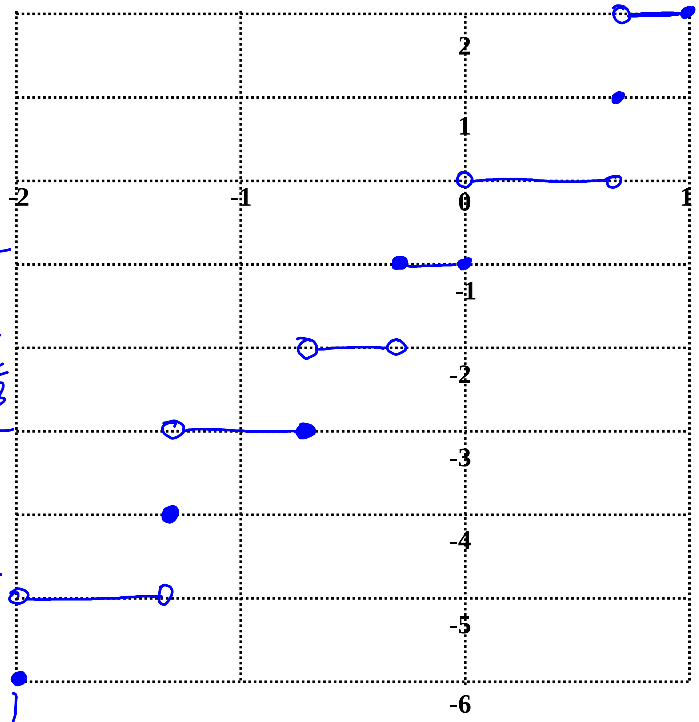
iv. Uncountable.

$A = (0,1)$
$B = (0,2)$

Interval of x	$\lceil . \rceil$	$\lfloor . \rfloor$	$\lceil . \rceil + \lfloor . \rfloor$
$\{-2\}$	-3	-3	-6
$(-2, -1\frac{1}{3})$	-2	-3	-5
$-1\frac{1}{3}$	-2	-2	-4
$(-1\frac{1}{3}, -\frac{2}{3})$	-1	-2	-3
$\{-\frac{2}{3}\}$	-1	-2	-3
$(-\frac{2}{3}, -\frac{1}{3})$	0	-2	-2
$[-\frac{1}{3}, 0]$	0	-1	-1
$(0, \frac{2}{3})$	1	-1	0
$\{2/3\}$	1	0	1
$(2/3, 1]$	2	0	2

b) (15 points) Draw the graph of the function

$$f(x) = \left\lceil \frac{3x}{2} \right\rceil + \left\lfloor x - \frac{2}{3} \right\rfloor \qquad \text{where } -2 \leq x \leq 1$$



$\lceil \frac{3x}{2} \rceil = -3 \iff -4 < \frac{3x}{2} \leq -3$
$-8 < 3x \leq -6$
$-\frac{8}{3} < x \leq -2$

$\lceil \frac{3x}{2} \rceil = -2 \iff -3 < \frac{3x}{2} \leq -2$
$-6 < 3x \leq -4$
$-2 < x \leq -\frac{4}{3}$

$\lceil \frac{3x}{2} \rceil = -1 \iff -2 < \frac{3x}{2} \leq -1$
$-4 < 3x \leq -2$
$-\frac{4}{3} < x \leq -\frac{2}{3}$

$\lceil \frac{3x}{2} \rceil = 0 \iff -1 < \frac{3x}{2} \leq 0$
$-2 < 3x \leq 0$
$-\frac{2}{3} < x \leq 0$

$\lceil \frac{3x}{2} \rceil = 1 \iff 0 < \frac{3x}{2} \leq 1$
$0 < 3x \leq 2$
$0 < x \leq \frac{2}{3}$

$\lceil \frac{3x}{2} \rceil = 2 \iff 1 < \frac{3x}{2} \leq 2$
$2 < 3x \leq 4$
$\frac{2}{3} < x \leq \frac{4}{3}$

$\lfloor x - \frac{2}{3} \rfloor = -2$
$-2 \leq x - \frac{2}{3} < -1$
$-1\frac{1}{3} \leq x < -\frac{1}{3}$

$\lfloor x - \frac{2}{3} \rfloor = 0$
$0 \leq x - \frac{2}{3} < 1$
$\frac{2}{3} \leq x < 1\frac{2}{3}$

$\lfloor x - \frac{2}{3} \rfloor = -3 \iff -3 \leq x - \frac{2}{3} < -2$
$-2\frac{1}{3} \leq x < -1\frac{1}{3}$

$\lfloor x - \frac{2}{3} \rfloor = -1$
$-1 \leq x - \frac{2}{3} < 0$
$-\frac{1}{3} \leq x < \frac{2}{3}$

Q2: [20 points] Answer the following questions.

a) (10 points) Determine whether the function $f(m,n) = m^3 - n^3$ where $f: Z \times Z \to Z$ is
   (i) 1:1,
   (ii) onto.
   Prove your answer.

(i) Not 1:1 $\quad f(1,1) = f(0,0) = 0$

(ii) Not onto

$f(0,0)=0$, $f(0,1)=-1$, $f(1,0)=1$,
$f(1,1)=0$, $f(1,2)=-7$, $f(2,1)=7$
$f(-1,0)=-1$, $f(0,-1)=1$, $f(-1,-1)=0$
$f(-1,-2)=7$,
$f(-1,2)=-9$,

It is obvious $\nexists$ no integers m,n s.t.
$m^3 - n^3 = 5 = (m-n)(m^2 + nm + n^2)$

b) (10 points) Show that the sequence $a_n = 7.2^n - n + 2$ is a solution to the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$.

$$a_{n-1} + 2a_{n-2} + 2n - 9 = 7.2^{n-1} - (n-1) + 2 + 2[7.2^{n-2} - (n-2) + 2] + 2n - 9$$
$$= 2^{n-1}[7+7] - n + 1 + 2 - 2n + 4 + 4 + 2n - 9$$
$$= 2^n(7) - n + 2$$
$$= a_n.$$

Q3: [10 points] Compute the following double summation:

$$\sum_{i=0}^{100}\sum_{j=10}^{20}\left(i+j\left(2^{i-1}\right)\right)$$

$$=\sum_{i=0}^{100}\sum_{j=10}^{20} i+\sum_{i=0}^{100}\sum_{j=10}^{20} j\left(2^{i-1}\right)$$

$$=\sum_{i=0}^{100} i\left(\sum_{j=10}^{20} 1\right)+\sum_{i=0}^{100} 2^{i-1}\left(\sum_{j=10}^{20} j\right)$$

$$=11\sum_{i=0}^{100} i+\sum_{i=0}^{100} 2^{i-1}\left(\sum_{j=1}^{20} j-\sum_{j=1}^{9} j\right)$$

$$=\frac{11(100)(101)}{2}+\sum_{i=0}^{100} 2^{i-1}\left(\frac{20(21)}{2}-\frac{9(10)}{2}\right)$$

$$=11(50)(101)+(210-45)\sum_{i=0}^{100} 2^{i-1}$$

$$=11(50)(101)+\frac{(210-45)}{2}\sum_{i=0}^{100} 2^{i}$$

$$=11(50)(101)+\frac{(210-45)}{2}\left(2^{100+1}-1\right)$$

Q4: [10 points] Use mathematical induction to prove that for all integers $n \geq 0$,

$$\left(1+\frac{1}{2}\right)^n \geq 1+\frac{n}{2}$$

Basis: $n=0$

$\left(1+\frac{1}{2}\right)^0 = 1 \geq 1+\frac{0}{2} = 1$ holds.

Induct^{n}: Assume that the result holds for n,

i.e. $\left(1+\frac{1}{2}\right)^n \geq 1+\frac{n}{2}$ $\forall n \geq 0$.

To show that $\left(1+\frac{1}{2}\right)^{n+1} \geq 1+\frac{n+1}{2}$.

By Induct^{n} hypothesis:

$$\left(1+\frac{1}{2}\right)^n \geq 1+\frac{n}{2}$$

multiply both sides by $\left(1+\frac{1}{2}\right)$, we get

$$\begin{aligned}
\left(1+\frac{1}{2}\right)^{n+1} &\geq \left(1+\frac{n}{2}\right)\left(\frac{3}{2}\right) \\
&= \frac{3}{2} + \frac{3n}{4} \\
&= 1 + \frac{3n+2}{4} \\
&= 1 + \frac{\frac{3n}{2}+1}{2} \\
&= 1 + \frac{n+1+\frac{n}{2}}{2} \\
&\geq 1 + \frac{n+1}{2} \quad \left(\frac{n}{2} \geq 0 \ \forall n \geq 0\right).
\end{aligned}$$

Q5: [15 points] Answer the following questions.

a) (5 points) Give a recursive definition of the set of positive integer powers of 3.

Basis: $3 \in S$.

Recursive step: $\forall x, y \in S$, $xy \in S$

b) (10 points) Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis Step: $(0,0) \in S$

Recursive step: If $(a,b) \in S$, then $(a, b+1) \in S$, $(a+1, b+1) \in S$ and $(a+2, b+1) \in S$. Use structural induction to show that $a \leq 2b$ whenever $(a,b) \in S$.

Basis: $0 \leq 2(0) = 0$ holds.

Inductive step: Let $(a,b) \in S$, where the property holds for (a,b). i.e. $a \leq 2b$

To show that the property is satisfied for $(a, b+1)$, $(a+1, b+1)$, $(a+2, b+1)$.

$(a, b+1)$: Since $a \leq 2b$ then

$a \leq 2b+2 = 2(b+1)$

$(a+1, b+1)$: Since $a \leq 2b$ then

$a+1 \leq 2b+1 \leq 2b+2 = 2(b+1)$

$(a+2, b+1)$: Since $a \leq 2b$ then,

$a+2 \leq 2b+2 = 2(b+1)$.

Q6: [20 points] Answer the following questions.

a) (5 points) A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

3^{50}

b) (15 points) In how many ways can a photographer arrange a family consisting of a father, a mother and 4 children in a row if

i. the mother must be next to the father?

$mf\ c_1 c_2 c_3 c_4 \to 5!$

$fm\ c_1 c_2 c_3 c_4 \to 5!$

Total $2(5!)$.

ii. the mother is not next to the father?

All arrangements − mother is next to father.

$= 6! - 2(5!)$.

iii. the mother is positioned somewhere to the left of the father?

$5+4+3+2+1$ different positions for the father & the mother →

```
fm----   f-m---   f--m-- .. f----m
-fm---   -f-m--   ...   -f---m
--fm--   ..   --f--m
---fm-   ---f-m
----fm
```

$$\sum_{i=1}^{5} i = \frac{5(6)}{2}.$$

$4!$ for the position of the children.

Total: $\frac{5(6)}{2} \cdot 4! = \frac{6!}{2}$.

Some Useful Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad , \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad , \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1} \text{ where } a \neq 1 \quad , \quad \sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \text{ where } |a|<1,$$

$$\sum_{i=0}^{n} ic^i = \sum_{i=1}^{n} ic^i = \frac{nc^{n+2} - nc^{n+1} - c^{n+1} + c}{(c-1)^2}$$

$$\sum_{i=1}^{\infty} ia^{i-1} = \frac{1}{(1-a)^2} \text{ where } |a|<1$$

$p \to (p \vee q)$	Addition	$[\neg q \wedge (p \to q)] \to \neg p$	Modus Tollens
$(p \wedge q) \to p$	Simplification	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$((p) \wedge (q)) \to (p \wedge q)$	Conjunction	$[(p \vee q) \wedge \neg p] \to q$	Disjunctive syllogism
$[p \wedge (p \to q)] \to q$	Modus Ponens	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$	Resolution

Some Useful Sequences	
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, 14641, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800
f_n Fibonacci	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …

$A \cap U = A$ $A \cup \Phi = A$	Identity Laws	$A \cup U = U$ $A \cap \Phi = \Phi$	Domination Laws
$A \cap A = A$ $A \cup A = A$	Idempotent Laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Laws
$\overline{(\bar{A})} = A$	Complementation Law	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative Laws
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ $\overline{A \cup B} = \bar{A} \cap \bar{B}$	De Morgan’s Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Laws	$A \cup \bar{A} = U$ $A \cap \bar{A} = \Phi$	Complement Laws